12.4 Cross Product.

Definition For

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \quad \text{and} \quad \vec{b} = \langle b_1, b_2, b_3 \rangle$$

The cross product of $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

or

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

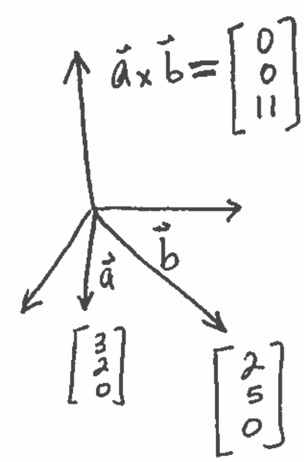
Find the cross product of

$$\vec{a} = \langle 3, 2, 0 \rangle \quad \text{and} \quad \vec{b} = \langle 2, 5, 0 \rangle$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} 2 & 0 \\ 5 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 3 & 2 \\ 2 & 5 \end{vmatrix} \hat{k}$$

$$= (0-0)\hat{\imath} - (0-0)\hat{\jmath} + (15-4)\hat{k}$$

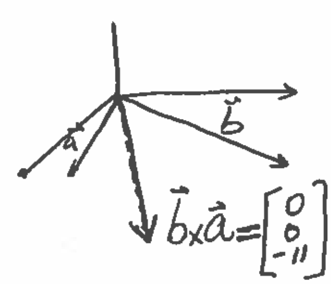
$$= 0\hat{\imath} + 0\hat{\jmath} + 11\hat{k} = 11\hat{k}$$



Also,

$$\vec{b} \times \vec{a} = \begin{vmatrix} 5 & 0 \\ 2 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} \hat{k}$$

$$= -11\hat{k}$$



Discuss "right-hand rule", and orthogonality

An alternative formula to express Cross product of $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ using a Symbolic determinant with the unit Vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ in its first row.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

<u>Theorem.</u> The vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

<u>Proof.</u> <u>Orthogonal to $\vec{a}$</u>, since

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= a_1 (a_2 b_3 - a_3 b_2) - a_2 (a_1 b_3 - a_3 b_1) + a_3 (a_1 b_2 - a_2 b_1)$$

$$= a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - a_3 a_2 b_1$$

$$= 0 \checkmark$$

<u>Theorem.</u> (Geometric description of the magnitude of $\vec{a} \times \vec{b}$)

The magnitude of $\vec{a} \times \vec{b}$ is given by

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta, \qquad 0 \le \theta \le \pi.$$

<u>Proof.</u>

$$|\vec{a} \times \vec{b}|^2 = (a_3 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$$

$$= a_2^2 b_3^2 + a_3^2 b_2^2 + a_3^2 b_1^2 + a_1^2 b_3^2 + a_1^2 b_2^2 + a_2^2 b_1^2$$
$$- 2 a_2 b_3 a_3 b_2 - 2 a_3 b_1 a_1 b_3 - 2 a_1 b_2 a_2 b_1$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

Turn for Lemma

$$= |\vec{a}|^2 \quad |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

$$= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta$$

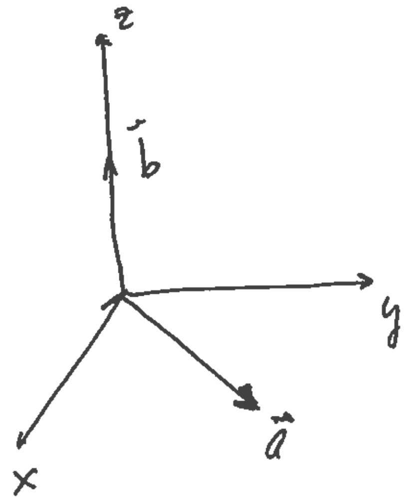
$$= |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta) = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta, \qquad 0 \le \theta \le \pi.$$

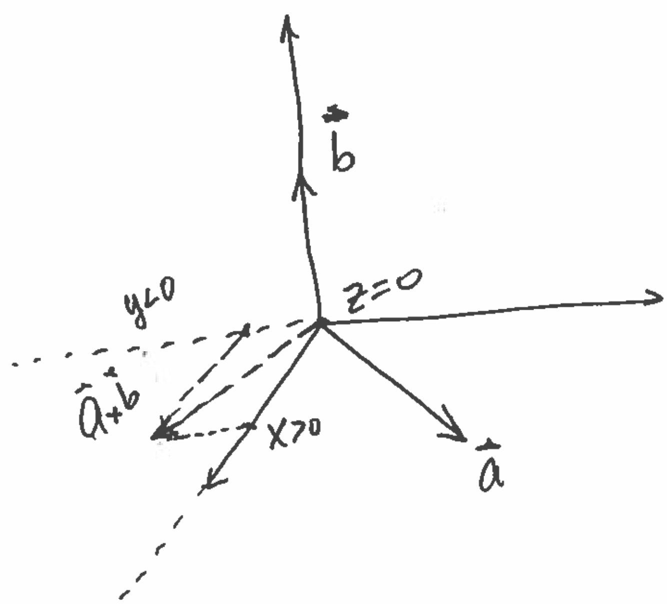
Section 12.4

Exercise #16

(Show slide).



$\vec{a} \times \vec{b} = ?$ Components are negative, positive, or zero.

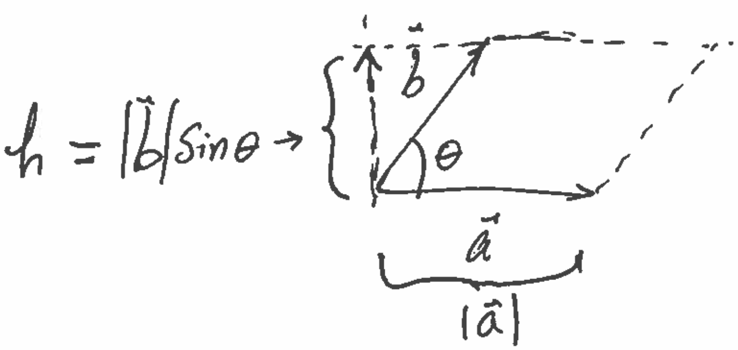


<u>Corollary</u> Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if

$$\vec{a} \times \vec{b} = \vec{0}$$

<u>Geometric interpretation of Cross product</u>

Consider the parallelogram formed by the two vectors $\vec{a}$ and $\vec{b}$.



Area = Base * height

or Area $= |\vec{a}||\vec{b}| \sin\theta$

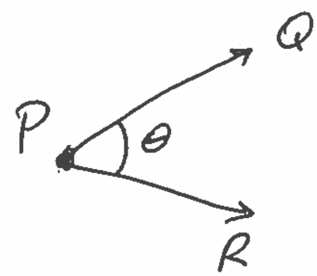
$= |\vec{a} \times \vec{b}|$

"The magnitude of the cross product $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$."

Exercise #29

29) Find a nonzero vector orthogonal to the plane through points $P(1,0,1)$, $Q(-2,1,3)$, $R(4,2,5)$.

Answ:



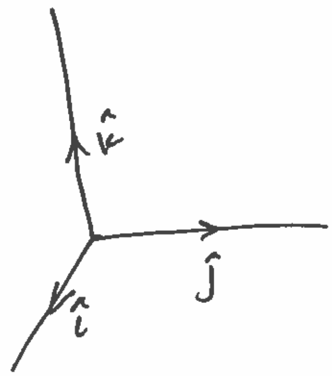
$(\vec{PQ}\times\vec{PR})$ is orthogonal to both $\vec{PQ}$ and $\vec{PR}$ So it's also orthogonal to the plane spanned by $\vec{PQ}$ and $\vec{PR}$.

$$\vec{PQ}\times\vec{PR} = \langle -3,1,2\rangle \times \langle 3,2,4\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} - \hat{j}\begin{vmatrix} -3 & 2 \\ 3 & 4 \end{vmatrix} + \hat{k}\begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 0\hat{i} - 18\hat{j} - 9\hat{k} = \langle 0, +18, -9\rangle.$$

Standard basis vectors: $\hat{i}, \hat{j}, \hat{k}$



$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

but

$$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i}, \quad \hat{i} \times \hat{k} = -\hat{j}$$

Is $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$?

Theorem 11 Properties of the cross product.

Slide

Triple Product

$\vec{a}\cdot(\vec{b}\times\vec{c}) =$

$\langle a_1, a_2, a_3\rangle \cdot \langle b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1\rangle$

$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Geometric Interpretation

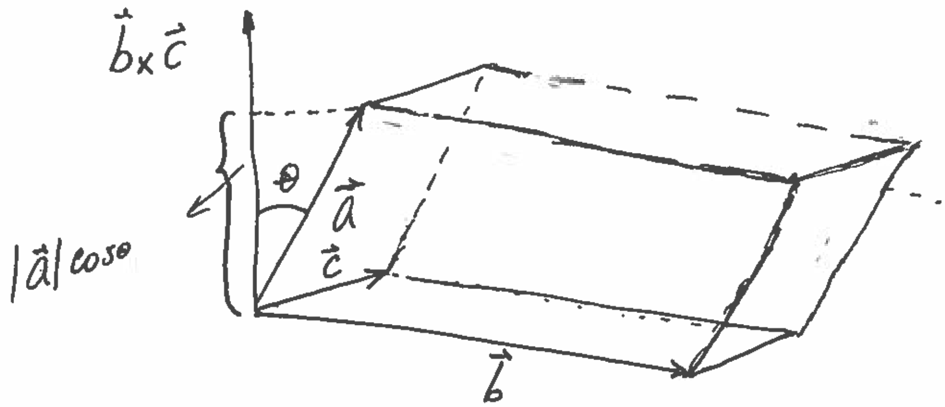
Slide

$\vec{a}\cdot(\vec{b}\times\vec{c}) = |\vec{a}|\,|\vec{b}\times\vec{c}|\cos\theta$

$\Rightarrow \quad |\vec{a}\cdot(\vec{b}\times\vec{c})| = |\vec{a}|\,|\vec{b}\times\vec{c}|\,|\cos\theta|$

Scalar triple product geometrical meaning.

Consider the parallelepiped formed by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.



$\theta = \measuredangle$ btw $\vec{a}$ and $(\vec{b}\times\vec{c})$

V = Volume of parallelepiped = Area of base * Height

$$= |\vec{b}\times\vec{c}|\,(|\vec{a}|\cos\theta)$$

$$= |\vec{a}|\,|\vec{b}\times\vec{c}|\,|\cos\theta| = |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

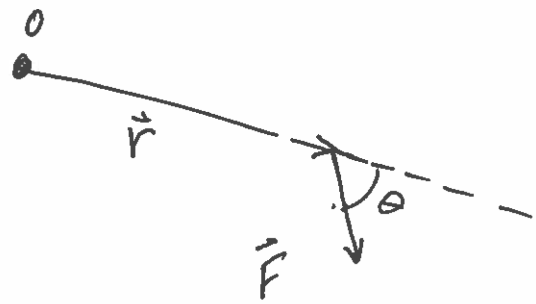
or

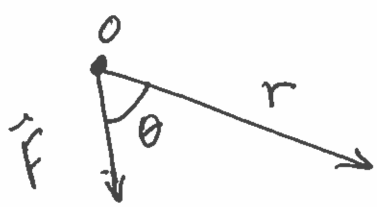
$$\boxed{V = |\vec{a}\cdot(\vec{b}\times\vec{c})|}$$

Remark: In the formula we use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \pi/2$.

Torque (Show slides)

Force $\vec{F}$ acting on a rigid body at a point given by a position vector $\vec{r}$.





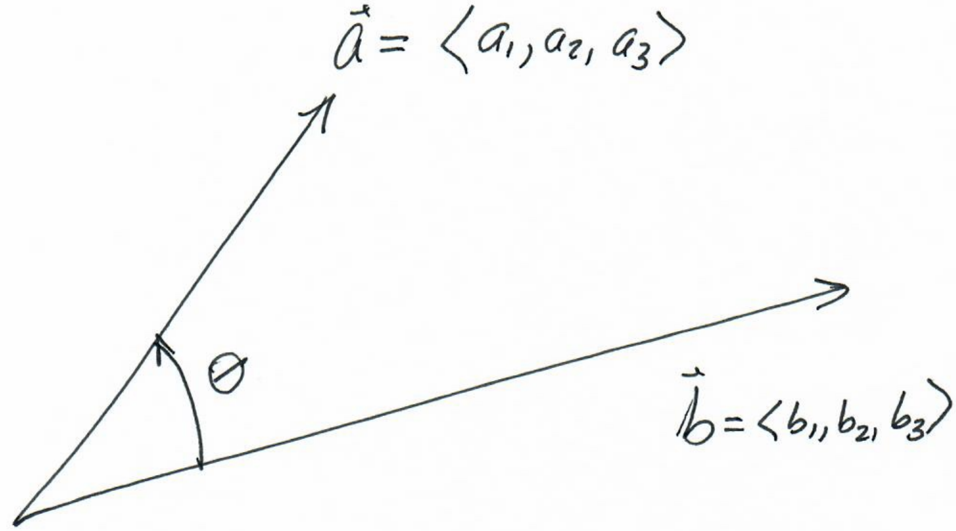
Definition

The torque τ (relative to the origin), is defined to be the cross product of the position and force vectors

$$\boxed{\tau = \vec{r} \times \vec{F}}$$

"tendency of the body to rotate about the origin".

$|\tau| = |\vec{r}|\,|\vec{F}| \sin\theta$



<u>Description of</u>

Dot Product : $\vec{a} \cdot \vec{b}$

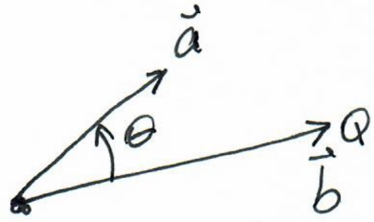
Cross Product : $\vec{a} \times \vec{b}$

Summary about Dot or Scalar Product and Cross or Vector Product.

	Dot Product	Cross Product
Algebraic Definition	$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$	$\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$
Geometrical Representation	$\vec{a}\cdot\vec{b} = \lvert\vec{a}\rvert\lvert\vec{b}\rvert\cos\theta$ $0 \le \theta \le \pi$	$\vec{a}\times\vec{b} = \lvert\vec{a}\times\vec{b}\rvert \begin{pmatrix}\text{rhs}\\ \text{rule}\end{pmatrix}\begin{pmatrix}\text{unit}\\ \text{vector}\end{pmatrix}$ where $\lvert\vec{a}\times\vec{b}\rvert = \lvert\vec{a}\rvert\lvert\vec{b}\rvert\sin\theta$ $0 \le \theta \le \pi$

Physical interpretation

Dot Product



$\vec{b} = \vec{PQ}$

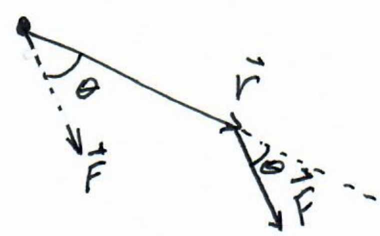
Work to go from P to Q $= \lvert\vec{b}\rvert \underbrace{\lvert\vec{a}\rvert\cos\theta}_{\text{Comp. of force } \vec{a} \text{ in the } \vec{b} \text{ direction}}$

$\lvert\vec{b}\rvert = \lvert\vec{PQ}\rvert$

$= \vec{a}\cdot\vec{b}$.

$0 \le \theta \le \pi$

Cross Product



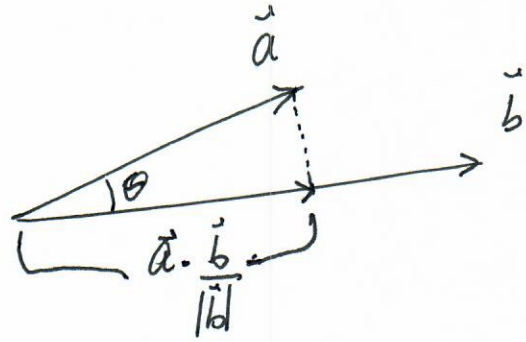
Torque $= \vec{\tau} = \vec{r}\times\vec{F}$

$\lvert\vec{\tau}\rvert = \lvert\vec{r}\rvert\lvert\vec{F}\rvert\sin\theta$

Dot Product

Geometrical Properties $\vec{a} \cdot \dfrac{\vec{b}}{|\vec{b}|} = \mathrm{comp}_{\vec{b}}\, \vec{a}$

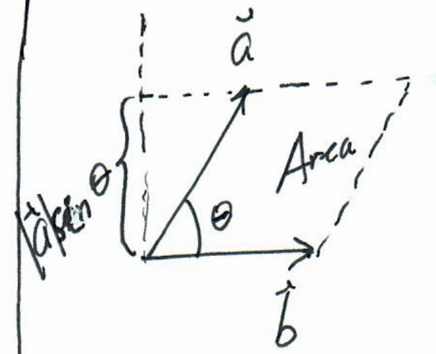
$= |\mathrm{proj}_{\vec{b}}\, \vec{a}|$



or $\vec{a} \cdot \dfrac{\vec{b}}{|\vec{b}|} = |\vec{a}| \cos\theta$

Since $\dfrac{a \cdot \vec{b}}{|\vec{b}|} = \dfrac{|\vec{a}|\,\cancel{|\vec{b}|}\cos\theta}{\cancel{|\vec{b}|}}$

Cross Product



$\text{Area} = |\vec{b}||\vec{a}|\sin\theta$

$= |\vec{a} \times \vec{b}|$